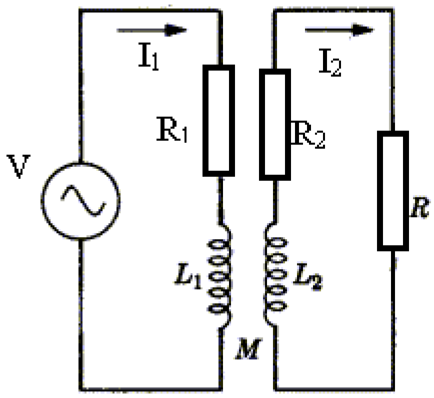
A mathematical study of two inductively coupled circuits shown below is presented here.



Let L_1 (L_2) be the inductance of the primary (secondary) coil and M be the mutal inductance between the two coils. The two coils themselves do not have resistance. Loop the primary coil to get

$$R_1 I_1 + L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V \qquad (1)$$

Similarly, for the secondary coil

$$R_S I_2 + L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = 0 \qquad (2)$$

where $R_S = R_2 + R$

If there is no flux leakage in the iron core,

$$L_1 = \alpha N_1^2,\ L_2 = \alpha N_2^2 \text{ and } M = \alpha N_1 N_2, \qquad (3)$$

where α is a constant depending on the geometrical factors of the two coils. Because of (3), we have

$$\frac{L_1 I_1 + M I_2}{N_1} = \frac{L_2 I_2 + M I_1}{N_2}, \qquad (4)$$

where N_1 (N_2) is the number of turns of the primary (secondary) coil.

Assume the a.c. source voltage varies with time in the form

$$V = V_o e^{i\omega t} \qquad (5)$$

With the complex amplitudes I_{1o} and I_{2o} ,the two currenrs can be written as

$$I_1 = I_{1o}e^{i\omega t} \quad (6i)$$

and

$$I_2 = I_{2o}e^{i\omega t} \quad (6ii)$$

By using (4), (5) and (6), (1) and (2) become

$$R_1I_{1o} + iX_1I_{1o} + iX_MI_{2o} = V_0 \quad (7)$$

$$R_SI_{2o} + iX_2I_{2o} + iX_MI_{1o} = 0 \quad (8)$$

where $X_1 = \omega L_1$, $X_2 = \omega L_2$ and $X_M = \omega M$.

Because of (3), $X_M^2 = X_1X_2$ (9)

From the above equations, we obtain the following results.

A. Primary current :

Solve I_{10} from (7) and (8), we get

$$(A + iB)I_{1o} = V_o$$ or

$$I_{1o} = \frac{V_o}{\sqrt{A^2 + B^2}}e^{i\phi}, \quad (10)$$

where $A = R_1 + \frac{X_m^2 R_S}{R_S^2 + X_2^2}$, $B = X_1 - \frac{X_2 X_m^2}{R_S^2 + X_2^2}$ and $\phi = -\tan^{-1}(\frac{B}{A})$

Note that $A^2 + B^2 = R_1^2 + X_1^2 + \frac{X_M^2(2R_1R_S - 1)}{R_S^2 + X_2^2}$.

When the resistive load in the secondary circuit is reduced, the primary current will increase.

B. Primary voltage:

The p.d. across the primary coil is $L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$ or

$$V_P = V - I_1 R_1$$
$$= (A - R_1 + iB) I_1$$
$$= \sqrt{(A - R_1)^2 + B^2} \; e^{i\phi'} \; I_1$$

where $\phi' = \tan^{-1}(\frac{B}{A - R_1}) = \tan^{-1}(\frac{R_S}{X_2})$ (11)

- **When R_S is infinite, $\phi' = \pi/2$. In other words, the primary voltage will lead the primary curent by $\pi/2$ when the secondary coil is open. A primary coil with no influence from the secondary coil is exactly identical to a pure inductor.**
- **When $R_S = 0$, $\phi' = 0$. In other words, the primary voltage and the primary curent are in phase.**

C. Secondary voltage:

The p.d. across the secondary coil, $V_S = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$

From V_P and (4), we obtain

$$\frac{V_P}{V_S} = \frac{N_1}{N_2} \quad (12)$$

It is true for all values of R_S.

D. Secondary current:

From (8),

$$I_{2o} = -\frac{iX_M}{R_S + iX_2} I_{10}$$

Only when $R_S = 0$, the ratio of the two currents satisfies

$$\frac{I_2}{I_1} = \frac{X_M}{X_2} = \frac{N_1}{N_2} \quad (13)$$

Reference: Principles of Electricity and Magnetism E.M. Pugh & E.W. Pugh
Addision Wesley,1960